

## Quantum fluctuations in FRLW space-time

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In this paper we study a quantum field theoretical approach, where a quantum probe is used to investigate the properties of generic non-flat FRLW space time. The fluctuations related to a massless conformal coupled scalar field defined on a space-time with horizon is identified with a probe and the procedure to measure the local temperature is presented.

*Keywords:* Quantum fluctuation; Temperature; Unruh effect.

### 1. Introduction

The Hawking radiation<sup>1</sup> is one of the most robust and important predictions of quantum field theory in curved space-time. Here we would like to study some (local) properties of a generic Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time with non-flat topology.

Let us remind some basic facts about the formalism. Any spherically symmetric four dimensional metric can be expressed in the form:

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + \mathcal{R}^2(x^i)d\Omega_2^2, \quad i, j \in \{0, 1\}, \quad (1)$$

with  $\gamma_{ij}(x^i)$  a tensor describing a two-dimensional space-time with coordinates  $x^i$ ,  $\mathcal{R}(x^i)$  being the “areal radius” and  $d\Omega^2$  encoding the metric of a two-dimensional sphere orthogonal respect to the first one.

The dynamical trapping horizon -if exists- is located in the correspondence of

$$\chi(x^i)|_H = 0, \quad \partial_i \chi(x^i)|_H \gtrless 0, \quad \chi(x^i) = \gamma^{ij}(x^i)\partial_i \mathcal{R}(x^i)\partial_j \mathcal{R}(x^i). \quad (2)$$

Thus, one may define the quasi-local Misner-Sharp gravitational energy as

$$E_{MS}(x^i) := \frac{1}{2G_N} \mathcal{R}(x^i) [1 - \chi(x^i)]. \quad (3)$$

For example, the mass of a black hole described by this formalism results to be  $E = \mathcal{R}_H/(2G_N)$ . The Killing vector fields  $\xi_\mu(x^\nu)$  are the generators of the isometries with  $\nabla_\mu \xi^\nu(x^\nu) + \nabla^\nu \xi_\mu(x^\nu) = 0$ : in the static case, with the time-like Killing vector field  $K^\mu = (1, 0, 0, 0)$ , the Killing surface gravity  $\kappa_K$  is given by

$$\kappa_K K^\mu(x^\nu) = K^\nu \nabla_\nu K^\mu(x^\nu). \quad (4)$$

In the dynamical case, the real geometric object which generalizes the Killing vector field is the Kodama vector field<sup>2</sup>,

$$\mathcal{K}^i(x^i) := \frac{1}{\sqrt{-\gamma}} \varepsilon^{ij} \partial_j \mathcal{R}(x^i), \quad i = 0, 1; \quad \mathcal{K}^i := 0, \quad i \neq 0, 1. \quad (5)$$

Thus, the Hayward surface gravity associated with dynamical horizon is<sup>3</sup>

$$\kappa_H := \frac{1}{2} \square_\gamma \mathcal{R}(x^i) \Big|_H. \quad (6)$$

The Hawking radiation is a thermal radiation of the black holes due to quantum effects. In the static case, all derivations of Hawking radiation lead to a semi-classical expression for the radiation rate  $\Gamma$  in terms of the exchange  $\Delta E_K$  of the Killing energy  $E_K$  and the Killing/Hawking temperature  $T_K$ ,

$$\Gamma \equiv e^{-\frac{2\pi\Delta E_K}{\kappa_K}}, \quad T_K := \frac{\kappa_K}{2\pi}. \quad (7)$$

In the dynamical case one may suggest the Kodama/Hayward temperature:

$$T_H := \frac{\kappa_H}{2\pi}. \quad (8)$$

An important example that demonstrates the covariance of the formalism is given by the de Sitter space-time. The static patch reads

$$ds^2 = -dt^2(1 - H_0^2 r^2) + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega^2, \quad (9)$$

where  $\mathcal{R} = r$  and the horizon is located at  $r_H = 1/H_0$  with surface gravity  $\kappa_H = H_0$ . The second patch is given by the expanding coordinates of the flat FLRW metric,

$$ds^2 = -dt^2 + e^{2H_0 t} (dr^2 + r^2 d\Omega^2), \quad (10)$$

where  $\mathcal{R} = e^{H_0 t} r$  and the dynamical (cosmological) horizon is  $r_H = 1/H_0$  with  $\kappa_H = H_0$ . Finally, the global patch in non-flat FLRW metric is given by

$$ds^2 = -dt^2 + \cosh^2(H_0 t) \left( \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega^2 \right), \quad (11)$$

with  $\mathcal{R} = r \cosh(H_0 t)$ , and  $r_H = 1/H_0$  and  $\kappa_H = H_0$  again. Now we will see how it is possible to associate a temperature to the dynamical horizon of flat and non-flat de Sitter space-time in (10)–(11).

## 2. Quantization of massless field in FLRW metric

We recall the quantization of a conformal coupled massless scalar field in the FLRW space-time. The metric reads

$$ds^2 = a^2(\eta)(-d\eta^2 + d\Sigma_3^2), \quad d\Sigma_3^2 = \frac{dr^2}{1 - kh_0^2 r^2} + r^2 dS_2^2. \quad (12)$$

where  $d\eta = dt/a(t)$  is the conformal time,  $h_0$  is a mass scale and the topology of the spacial section can be flat, spherically or hyperbolic for  $k = 0, 1, -1$ , respectively.

Given a massless scalar field,

$$\phi(x) = \sum_{\alpha} f_{\alpha}(x) a_{\alpha} + f_{\alpha}^*(x) a_{\alpha}^{\dagger}, \quad (13)$$

such that the modes are conformal invariant, namely  $(\square - R/6)f_\alpha(x) = 0$ , the associated Wightman function  $W(x, x') = \langle \phi(x)\phi(x') \rangle$  results to be

$$W(x, x') = \sum_\alpha f_\alpha(x)f_\alpha^*(x'), \quad \left(\square - \frac{\mathcal{R}}{6}\right)W(x, x') = 0. \quad (14)$$

The Wightman function satisfies the following rule for the conformal transformations of the metric:

$$ds^2 = \Omega(x)^2 ds_0^2, \quad \phi = \frac{1}{\Omega}\phi_0, \quad W(x, x') = \frac{1}{\Omega(x)\Omega(x')}W_0(x, x'). \quad (15)$$

We may also take  $W(x, x') = W(\eta - \eta', r - r')$  due to the homogeneity and isotropy of FLRW space-times.

Let us consider the spherical case ( $k = 1$ ) in (12),

$$ds^2 = a^2(\eta) \left( -d\eta^2 + d\chi^2 + \frac{1}{h_0^2} \sin^2 h_0 \chi dS_2^2 \right), \quad h_0 \chi = \arcsin h_0 r. \quad (16)$$

This metric is conformally related to the Minkowski space-time,

$$ds^2 = a^2(\eta) 4 \cos^2 \left( h_0 \frac{\eta + \chi}{2} \right) 4 \cos^2 \left( h_0 \frac{\eta - \chi}{2} \right) (-dt^2 + dr^2 + r^2 dS_2^2), \quad (17)$$

with

$$t \pm r = \frac{1}{h_0} \tan \left( h_0 \frac{\eta \pm \chi}{2} \right). \quad (18)$$

Thus, by starting from the well-known Wightman function in Minkowski space-time, one can use (15) and derive for the spherical FLRW metric

$$W(x, x') = \frac{h_0^2}{8\pi^2 a(\eta)a(\eta')} \frac{1}{\cos(h_0(\eta - \eta')) - \cos(h_0(\chi - \chi'))}. \quad (19)$$

The hyperbolic case  $k = -1$  is obtained with the substitution  $h_0 \rightarrow ih_0$ , while the flat case  $k = 0$  corresponds to the limit  $h_0 \rightarrow 0$ .

### 3. Quantum fluctuations in flat space-time

Let us consider a free massless quantum scalar field  $\phi(x)$  in thermal equilibrium at the temperature  $T$  in flat space-time. We know that finite temperature field theory effects of this kind can be investigated by given that the scalar field defined in the Euclidean manifold  $S_1 \times R^3$ , where the imaginary time is  $\tau = -it$ , compactified in the circle  $S_1$  with period  $\beta = 1/T$ .

We briefly review the local quantity  $\langle \phi(x)^2 \rangle$ , which is a divergent quantity due to the product of valued operator distribution in the same point  $x$ . By making use of the zeta-function regularization procedure, the quantum fluctuations read<sup>4,5</sup>:

$$\langle \phi(x)^2 \rangle = \zeta(1|L_\beta)(x), \quad L_\beta = -\partial_\tau^2 - \nabla^2, \quad (20)$$

4

where  $\zeta(z|L_\beta)(x)$  is the local zeta-function associated with the operator  $L_\beta$ . It is easy to see that the analytic continuation of the local zeta-function is regular at  $z = 1$  and finally one gets

$$\langle \phi(x)^2 \rangle = \frac{1}{12\beta^2} = \frac{T^2}{12}. \quad (21)$$

In this way we obtain the temperature of the quantum field in thermal equilibrium from the zeta-function renormalized vacuum expectation value, namely we have a quantum thermometer.

### 3.1. Quantum fluctuations in FLRW space-time

Now we extend the argument to generic FLRW metric. The off-diagonal Wightman function (19) leads to

$$W(x, x') = \langle \phi(x)\phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{\Sigma^2(x, x')}, \quad (22)$$

with

$$\Sigma^2(\tau, \tau - s) = a(\tau)a(\tau - s) \frac{2}{h_0^2} (\cos h_0(\Delta\chi(s)) - \cos h_0(\Delta\eta(s))), \quad (23)$$

where  $a(\tau)$  is the conformal factor. Thus, in the limit  $s \rightarrow 0$ , one has

$$\langle \phi(x)^2 \rangle = W(\tau, \tau). \quad (24)$$

It is possible to show that

$$W(\tau, \tau - s) = -\frac{1}{4\pi s^2} + \frac{B}{48\pi^2} + O(s^2), \quad (25)$$

where

$$B = H^2 + A^2 + 2\dot{H}t + \frac{h_0^2}{a^2}(1 - 2\dot{t}^2), \quad A^2 = \frac{1}{\dot{t}^2 - 1} (\ddot{t} + H(\dot{t}^2 - 1)), \quad (26)$$

the dot being the derivative respect to the proper time,  $H = (da(t)/dt)/a(t)$  being the usual Hubble parameter, and  $A^2$  the radial acceleration. Therefore, after the regularization for the divergent part at  $s \rightarrow 0$ ,

$$\langle \phi^2 \rangle_R = \frac{1}{48\pi^2} \left( H^2 + A^2 + 2\dot{H}t \pm \frac{h_0^2}{a^2}(1 - 2\dot{t}^2) \right). \quad (27)$$

This result is quite general and it is valid also for spatial curvature  $k \neq 0$ .

### 3.2. Quantum fluctuations in non flat de Sitter space-time

In the case of de Sitter space-time with  $k = 1$ , we may put  $H_0 = h_0$  and the expression for quantum fluctuations reads

$$\langle \phi^2 \rangle_R = \frac{1}{48} (H_0^2 + A^2) = \frac{1}{48\pi^2} \frac{H_0^2}{1 - R_0^2 H_0^2}, \quad (28)$$

where  $R_0 = \text{const}$  is the areal radius of the Kodama observer and the acceleration has been computed as

$$A^2 = \frac{R_0^2 H_0^4}{1 - R_0^2 H_0^2}. \quad (29)$$

For a Kodama observer with  $R_0 = 0$  we recover the Gibbons-Hawking temperature associated with de Sitter space-time,

$$T = \frac{H_0}{2\pi}. \quad (30)$$

This is an important check of our approach, since it shows the coordinate independence of the result for the important case of de Sitter space-time.

### 3.3. Quantum fluctuations in FRLW form of Minkowski space-time

The Minkowski space-time may be written in a FRLW form with hyperbolic section  $k = -1$  (Milne universe),

$$ds_M^2 = -dt^2 + t^2 \left( \frac{dr^2}{1+r^2} + r^2 d\Omega_2^2 \right), \quad h_0 = 1. \quad (31)$$

Making use of the Hayward formalism, it is easy to verify that there is no dynamical horizon and the surface gravity is vanishing. In this case we obtain

$$\langle \phi^2 \rangle_R = \frac{A^2}{48\pi^2}, \quad (32)$$

namely only the radial acceleration  $A^2$  is present and the temperature is defined as

$$T_U = \frac{A}{2\pi}, \quad (33)$$

recovering the well known Unruh effect.

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